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TRANSIENT RADIAL TEMPERATURE DISTRIBUTIONS IN CYLINDRICAL SHELLS

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INTRODUCTION

BRUIN and Beverloo [1] have treated the problem of the transient response of a hollow cylinder to a step in flux at the inner surface.† This problem was considered concurrently by Gandhi [2] and by Ölçer and Sunderland [3]. Gandhi corroberates the results of Bruin–Beverloo. Ölçer and Sunderland treat a general problem which can be specialized to the problem of Bruin–Beverloo and Gandhi.‡ A related problem, the transient response of a hollow cylinder to a step in flux at the outer surface, was previously discussed by Phythian [5]; this problem is also a specialization of Ölçer and Sunderland's problem. The transient response of a hollow cylinder to arbitrary, time-dependent heat fluxes at the inner and outer surfaces can be calculated by using Duhamel's theorem ([6] pp. 30–32) and the solutions of Bruin–Beverloo and Phythian.

Bruin and Beverloo have thoroughly discussed the solution appropriate for large values of Fo, the dimensionless time. This solution is impractical for calculations when $Fo \to 0$ because the infinite summation converges very slowly. In certain cases ($Fo \to 0$, $\lambda \gg 1$) the truncation error can be appreciable even when using fifty§ terms in the summation. One purpose of this note is to present a solution which is appropriate for small values of Fo; this solution has not been discussed by previous authors. A second purpose of this note is to present the results in a graphical form convenient for rapid, hand calculations.

SOLUTION USEFUL FOR SMALL VALUES OF F_0

A solution useful for small values of Fo was obtained by using the procedure outlined by Carslaw and Jaeger ([6] pp. 330–331). Equation (5) of [1] was expanded using the binomial theorem and the asymptotic expansions of the Bessel functions. The solution is

$$\theta = 2\sqrt{\frac{Fo}{R}}\Bigg\{\Bigg\{\sum_{n=0}^{\infty}\Bigg\{\Bigg[\mathrm{ierfc}(f) - \frac{1+3R}{4R}Fo^{\frac{1}{2}}\, i^2\mathrm{erfc}(f) + \frac{9+6R+33R^2}{32R^2}Fo\, i^3\mathrm{erfc}(f) - \frac{75+27R+33R^2+249R^3}{128R^3}Fo^{\frac{3}{2}}\, i^4\mathrm{erfc}(f) + \ldots\Bigg]$$
$$+\Bigg[\mathrm{ierfc}(g) + \frac{1-3R}{4R}Fo^{\frac{1}{2}}\, i^2\mathrm{erfc}(g) + \frac{9-6R+33R^2}{32R^2}Fo\, i^3\mathrm{erfc}(g) + \frac{75-27R+33R^2-249R^3}{128R^3}Fo^{\frac{3}{2}}\, i^4\mathrm{erfc}(g) + \ldots\Bigg]\Bigg\} + \frac{3}{2}\frac{\lambda-1}{\lambda}\sum_{n=0}^{\infty}\Bigg[\Bigg[n\Bigg\{\Bigg[Fo^{\frac{1}{2}}\, i^2\mathrm{erfc}(f)$$
$$- \frac{1+3R}{4R}Fo\, i^3\mathrm{erfc}(f) + \frac{9+6R+33R^2}{32R^2}Fo^{\frac{3}{2}}\, i^4\mathrm{erfc}(f) - \ldots\Bigg] + \Bigg[Fo^{\frac{1}{2}}\, i^2\mathrm{erfc}(g) + \frac{1-3R}{4R}Fo\, i^3\mathrm{erfc}(g)$$
$$+ \frac{9-6R+33R^2}{32R^2}Fo^{\frac{3}{2}}\, i^4\mathrm{erfc}(g) + \ldots\Bigg]\Bigg\}\Bigg]\Bigg] + \ldots\Bigg\}.$$

where

$$f = f(n) = \frac{2n(\lambda-1) + (R-1)}{2\sqrt{Fo}}$$
$$g = g(n) = \frac{2(n+1)(\lambda-1) - (R-1)}{2\sqrt{Fo}}. \qquad (1)$$

For practical calculations one truncates the solution. To illustrate the effect of truncation, three solutions are considered.

$$\theta_2^* = 2\sqrt{\left(\frac{Fo}{R}\right)}\left[\mathrm{ierfc}(f) - \frac{1+3R}{4R}Fo^{\frac{1}{2}}\, i^2\mathrm{erfc}(f)\right]$$

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† For nomenclature and a complete statement of the problem see [1]. Two typographical errors appear in [1]:

(1) Equation (4) should read "$\partial\theta/\partial R = -1$; $R = 1$; $\partial\theta/\partial R = 0$, $R = \lambda$".

(2) The exponential appearing in the definition of the ordinate of Fig. 5 should read "$\exp(\mu_n^2 Fo)$".

‡ Ölçer [4] has also solved the general problem of two concentric layers in imperfect thermal contact.

§ The number used by Bruin and Beverloo.

* This solution is also the truncated solution discussed by Carslaw and Jaeger ([6] p. 339) for the region bounded internally by a circular cylinder.

$$\theta_4 = \theta_2 + 2\sqrt{\left(\frac{Fo}{R}\right)}\left[\frac{9 + 6R + 33R^2}{32R^2} Fo\, i^3\text{erfc}(f) + i\text{erfc}(g)\right]$$

$$\theta_8 = \theta_4 + 2\sqrt{\left(\frac{Fo}{R}\right)}\left[-\frac{75 + 27R + 33R^2 + 249R^3}{128R^3} \times Fo^{3/2}\, i^4\text{erfc}(f) + \frac{1 - 3R}{4R} Fo^{1/2}\, i^2\text{erfc}(g) + \frac{9 - 6R + 33R^2}{32R^2} Fo\, i^3\text{erfc}(g) + \frac{75 - 27R + 33R^2 - 249R^3}{128R^3} Fo^{3/2}\, i^4\text{erfc}(g)\right],$$

where

$$f = \frac{R - 1}{2\sqrt{Fo}}$$

$$g = \frac{2(\lambda - 1) + (R - 1)}{2\sqrt{Fo}}.$$

Following Bruin and Beverloo, the truncation errors are defined as

$$\delta_2 = \left|\frac{\theta_{50\,\text{terms}} - \theta_2}{\theta_{50\,\text{terms}}}\right|.100$$

$$\delta_4 = \left|\frac{\theta_{50\,\text{terms}} - \theta_4}{\theta_{50\,\text{terms}}}\right|.100$$

$$\delta_8 = \left|\frac{\theta_{50\,\text{terms}} - \theta_8}{\theta_{50\,\text{terms}}}\right|.100$$

$\theta_{50\,\text{terms}}$ represents the Bruin–Beverloo solution truncated after fifty terms. Figure 1 presents the truncation error for a shell with thickness of 3/2 the internal radius ($\lambda = 5/2$); the truncation errors for the new solution are compared with those for the Bruin–Beverloo solution truncated after five terms (Δ_5) and after one term (Δ_1). The range of Fo for which equation (1) or its truncated forms is useful will increase with increasing λ.

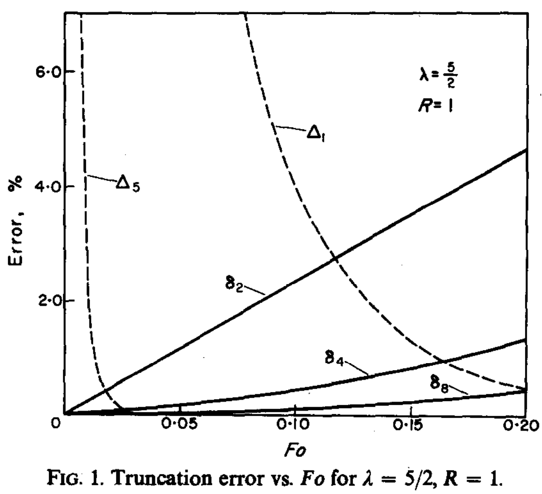

FIG. 1. Truncation error vs. Fo for $\lambda = 5/2$, $R = 1$.

GRAPHICAL RESULTS

Bruin and Beverloo selected the dimensionless variables most convenient for analyzing the problem. These variables also were appropriate for obtaining the solution for small values of Fo; however they are not suitable for the graphical results. The following "primed" dimensionless variables are used to present the graphical results.

$$\lambda' \equiv \frac{R_i}{R_0 - R_i} = \frac{\text{internal radius}}{\text{shell thickness}} = \frac{1}{\lambda - 1}$$

$$\theta' \equiv \frac{2\pi k(T - T_0)R_i}{Q(R_0 - R_i)} = \theta\lambda'$$

$$Fo' \equiv \frac{\alpha t}{(R_0 - R_i)^2} = Fo(\lambda')^2$$

$$R' \equiv \frac{r - R_i}{R_0 - R_i} = (R - 1)\lambda'.$$

$\lambda' = \infty$ represents the plane slab subjected to a step in flux.

Following the form of Carslaw and Jaeger ([6] p. 113, p. 203, p. 242), the temperature is transformed further by subtracting the linear increase with time; this transformation permits the presentation of results for all values of Fo' rather than for a limited range. The graphical results are presented for four values of λ' (4, 3/2, 2/3, 1/3) in Figs. 2–5.

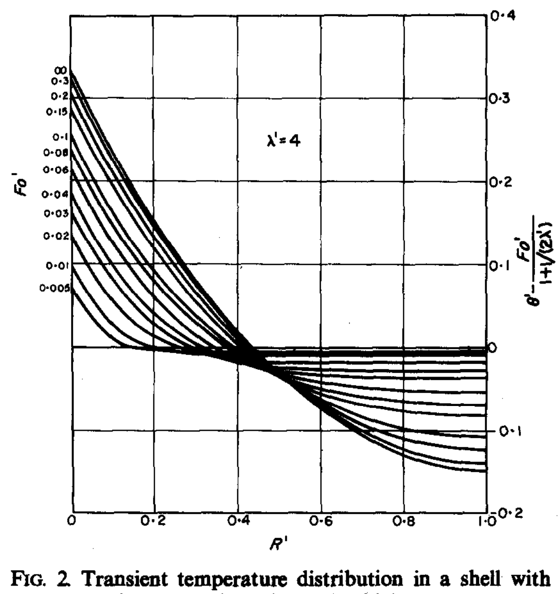

FIG. 2. Transient temperature distribution in a shell with internal radius of 4 × the thickness.

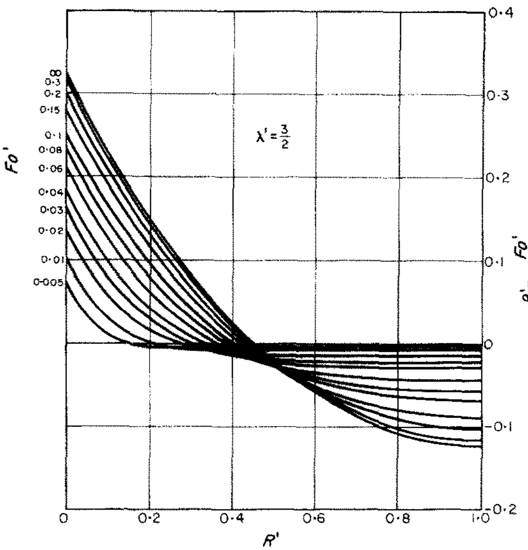

FIG. 3. Transient temperature distribution in a shell with internal radius of 3/2 × the thickness.

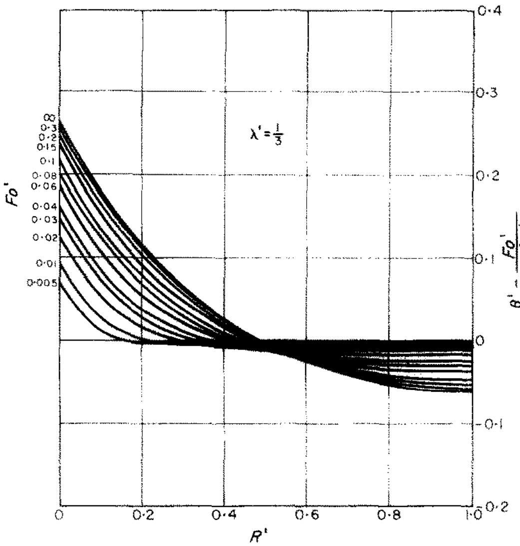

FIG. 5. Transient temperature distribution in a shell with internal radius of 1/3 × the thickness.

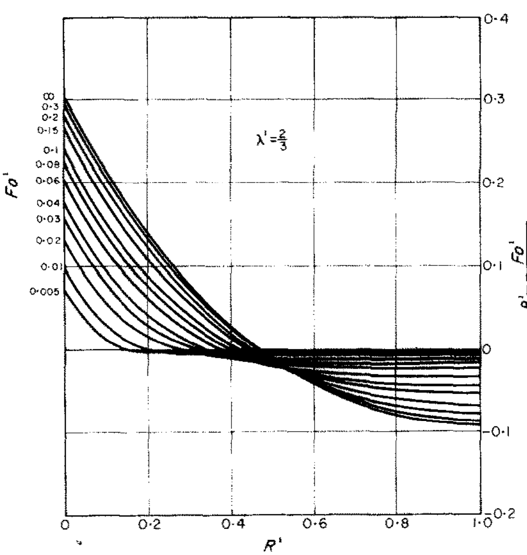

FIG. 4. Transient temperature distribution in a shell with internal radius of 2/3 × the thickness.

The solution for $\lambda' = \infty$ is given in Carslaw and Jaeger ([6] p. 113) and differs only slightly from Fig. 2. To obtain solutions for other values of λ', one may interpolate between the figures for $1/3 \leqslant \lambda' \leqslant \infty$.

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